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Algebra in Integral domain and field.

Q.1. Define Integral domain and field given an example.

Soln. Integral domain: → A commutative ring ~~with~~ R with unit element having no zero divisors is called an Integral domain.

For example: The ring of integer $(\mathbb{Z}, +, \cdot)$ is an integral domain.

Field: → A commutative ring R with unit element having at least two elements is called a field if every non-zero elements of R possesses their multiplicative inverse.

For example: The ring of rational numbers $(Q, +, *)$ is a field.

Q.2. Prove that every field is an integral domain.

proof: let F be a field. We have to show that F is an integral domain. Since, F is a field so it is commutative ring with unit element. Therefore, in order to show F to be an integral domain, we ~~have~~ only have to prove that F has no zero divisors. For this

Let $a \in F$ and $a \neq 0$, then a^{-1} exists in F. We have

$ab = 0 \Rightarrow a^{-1}(ab) = a^{-1}0 \Rightarrow (a^{-1}a)b = a^{-1}0$ [by associative law]

$\Rightarrow 1b = a^{-1}0$ [$\because a^{-1}a = 1$] $\Rightarrow 1b = 0$ [By elementry property of ring]

$\Rightarrow b = 0$ [$\because 1.a = a = a.1$]

~~$b \neq 0$~~

Similarly, let $b \in F$ and $b \neq 0$ and

$ab = 0 \Rightarrow (ab)b^{-1} = 0b^{-1}$

$\Rightarrow a(bb^{-1}) = 0$ [$\because 0b^{-1} = 0 b^{-1} 0$]

$\Rightarrow a1 = 0$ [$\because bb^{-1} = 1$]

$\Rightarrow a = 0$ [$\because a.1 = a = 1.a$]

Thus, we obtained that in F, $ab = 0$, then either $a = 0$ or $b = 0$

Hence, F is an integral domain. proved.

Q.3. Prove that every finite integral domain is a field.

Proof: Let D be a finite integral domain. Therefore, by the definition of integral domain we have that D is a commutative ring with unit element having no zero divisors.

Let $D = \{a_1, a_2, \cdots a_n\}$. In order to prove that D is a field we only have to prove that D has multiplicative inverse for every non-zero element in D. Let $a \neq 0$ be any arbitrary element of D and consider the set

$$D_1 = \{aa_1, aa_2, \cdots aa_n\}$$

D_1 has n distinct products. Let $aa_i = aa_j$ [$\because i \neq j$]

$\Rightarrow a(a_i - a_j) = 0$ [by left distributive law]

Since, D has no zero divisors and $a \neq 0$, then

$$a_i - a_j = 0 \quad \text{for } i \neq j$$

$$\Rightarrow a_i = a_j \quad \text{for } i \neq j$$

$\Rightarrow$ This is a contradiction, because D has n distinct elements $a_1, a_2, \cdots a_n$.

Consequently, D_1 has n distinct products. But the elements of D_1 are the elements of D placed in some order. Further, D has unit element, that is.

$$1 \in D$$

$$\Rightarrow 1 \in D_1$$

This implies that there exists an element b in D such that

$$ab = 1$$

But D is commutative. Therefore

$$ab = 1 = ba$$

Thus a^{-1} exists in D. Hence, D is a field.

Proved.